The reader may similarly show

$$
\begin{aligned}
\ln 4 &= \int_0^1 \frac{1+2x+3x^2}{1+x+x^2+x^3}\,dx \\
&= \left(1+\frac{1}{2}+\frac{1}{3}-\frac{3}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}-\frac{3}{8}\right)+\cdots \\
&\quad+\left(\frac{1}{4k-3}+\frac{1}{4k-2}+\frac{1}{4k-1}-\frac{3}{4k}\right) \\
&\quad+\int_0^1 \frac{x^{4k}+2x^{4k+1}+3x^{4k+2}}{1+x+x^2+x^3}\,dx \\
&= 1+\frac{1}{2}+\frac{1}{3}-\frac{3}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}-\frac{3}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}-\frac{3}{12}+\cdots .
\end{aligned}
$$

it is straightforward though cumbersome to show

$$
\begin{aligned}
\ln N = 1 &+ \frac{1}{2}+\cdots+\frac{1}{N-1}-\frac{N-1}{N}+\frac{1}{N+1}+\frac{1}{N+2}+\cdots \\
&+\frac{1}{2N-1}-\frac{N-1}{2N}+\cdots .
\end{aligned}
$$

References

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Simple Geometric Solutions to De l'Hospital's Pulley Problem

Raymond Boute, INTEC, University of Ghent, Belgium

In a recent paper, Hahn [2] discusses De l'Hospital's solution to the following problem. A weight is attached by a cord to a point in the ceiling. The cord runs over a pulley attached by a cord of length r to a point in the ceiling at distance d from the first as shown in Figure 1. The problem is to find the equilibrium configuration,

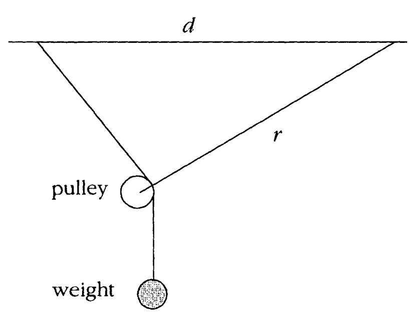

Figure 1. Arrangement of weight and pulley.

assuming an ideal weightless pulley with negligible radius and ideal, weightless cords.

De l'Hospital's solution is by calculus, based on minimal potential energy. We propose solutions by elementary geometry, based on the balance of forces acting on the pulley.

Assume $r < d$, the case $r \geq d$ being trivial. Since the tension in the cord carrying the weight is constant, the equilibrium of forces acting on the pulley requires $\beta = \gamma$ in the Figure 2(a). Clearly, $\gamma = \pi/2 + \alpha$ and hence $\beta = \pi/2 + \alpha$. Let us subdivide β by line segments r' and x' into angles α, α and $\frac{\pi}{2} - \alpha$ respectively, as shown

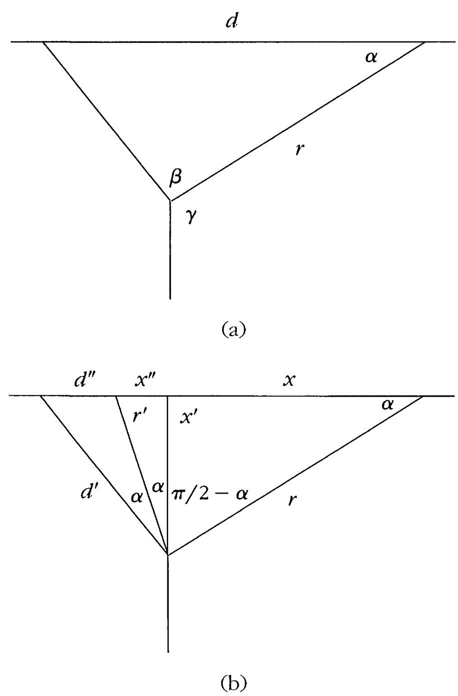

Figure 2. Geometric approach by ratio chaining.

in the Figure 2(b), thereby also subdividing d into d'', x'' and x. Triangles (d'', d', r') and (d', d, r) are obviously similar, and so are (r', x', x'') and (r, x, x'); hence

$$\frac{d''}{d'} = \frac{d'}{d} = \frac{r'}{r} = \frac{x'}{x} = \frac{x''}{x'} = a, \quad \text{which defines } a.$$

We call this technique *ratio chaining* for obvious reasons. Now $d'' = a \cdot d' = a^2 \cdot d$ and $x'' = a \cdot x' = a^2 \cdot x$, hence $d = d'' + x'' + x = a^2 \cdot (d + x) + x$ or, with $a^2 = \left(\frac{x'}{x}\right)^2 = \frac{r^2 - x^2}{x^2}$,

$$d = \frac{r^2 - x^2}{x^2} \cdot (d + x) + x.$$

Simplification yields $2 \cdot d \cdot x^2 - r^2 \cdot x - r^2 \cdot d = 0$, as in **[2]**.

As an aside, it is instructive to write the equation in dimensionless form:

$$2\cdot\left(\frac{x}{r}\right)^2-\frac{r}{d}\cdot\left(\frac{x}{r}\right)-1=0.$$

The nonnegative solution is

$$\cos\alpha=\frac{x}{r}=\frac{1}{4}\cdot\left(\frac{r}{d}+\sqrt{\left(\frac{r}{d}\right)^2+8}\,.\right)$$

Recalling $\frac{r}{d}\leq 1$, the limiting case $\frac{r}{d}=1$ provides a quick verification ($x=r$).

Of course, the problem can also be solved using brute force trigonometry. By inspection, $x'=r\cdot\sin\alpha$ and $d-r\cdot\cos\alpha=x'\cdot\tan(2\cdot\alpha)$, hence

$$d-r\cdot\cos\alpha=r\cdot\sin\alpha\cdot\tan(2\cdot\alpha)$$

By elaborating $\tan(2\cdot\alpha)$, this becomes $d/r=\cos\alpha+\sin\alpha\cdot\frac{2\cdot\sin\alpha\cdot\cos\alpha}{(\cos\alpha)^2-(\sin\alpha)^2}$. With $(\sin\alpha)^2=1-(\cos\alpha)^2$, this can be simplified to $d/r=\frac{\cos\alpha}{2\cdot(\cos\alpha)^2-1}$. Substituting x/r for $\cos\alpha$ again yields the desired equation.

Finally, we recall a crucial rule of thumb in physics and geometry: *where symmetry exists, try exploiting it.* The cord r holding the pulley is the symmetry axis for the configuration of the cord carrying the weight. Connecting the attachment point of the latter to its mirror image (which lies on the vertical through the pulley) yields the basis b of an isosceles triangle with height h, as shown in Figure 3.

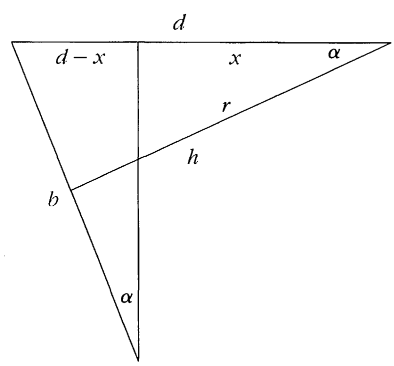

Figure 3. Geometric approach exploiting symmetry.

By triangle similarity, $\frac{x}{r}=\frac{h}{d}$ and $\frac{b}{d-x}=\frac{d}{b/2}$. This yields $\left(\frac{x}{r}\right)^2=\frac{d^2-(b/2)^2}{d^2}=1-\frac{1}{4}\cdot\left(\frac{b}{d}\right)^2$ and $\left(\frac{b}{d}\right)^2=2\cdot\left(1-\frac{x}{d}\right)$ respectively, and hence $\left(\frac{x}{r}\right)^2=\frac{1}{2}\cdot\left(1+\frac{x}{d}\right)$, the desired equation.

To a physicist, an argument based on pure statics may be aesthetically more pleasing than one based on energy considerations. Indeed, in *Lectures on Physics*, Richard Feynman gives at one point a derivation about forces using the conservation of energy principle, followed by the remark that Simon Stevin (Flemish engineer and mathematician, 1548–1620) provided a superior and truly brilliant derivation using just statics. Feynman goes on by mentioning that the picture representing this derivation appears on Stevin's tombstone as an epitaph, and concludes: "If you have such an epitaph on your tombstone, you're doing fine."

Regarding the pulley problem, the existence of a geometric solution based on statics provides reason for speculating that Archimedes could have solved this problem, and perhaps did solve similar ones. Indeed, although Archimedes did not have vectors as we know them for decomposing forces, his work on tackles certainly reflects sufficient insight, if not for tackling the general case, then certainly for cases of this particular kind where pulleys induce symmetry. The book *La Rivoluzione Dimenticata* by Lucio Russo, as presented in a recent review by Sandro Graffi **[1]**, certainly provides food for thought about how much scientific knowledge was available in antiquity.

References

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The Derivative of Sin θ

Selvaratnam Sridharma (ssridharma@dillard.edu), Dillard University, New Orleans, LA 70122

An unusual argument supporting the derivative formula for $\sin\theta$ can be offered using Figure 1.

Since the distance from P to Q is $2\sin(\Delta\theta/2)$ (when Δy and $\Delta\theta$ are positive quantities as in the figure), we have $\sin\phi = \Delta y/2\sin(\Delta\theta/2)$. Also, we see that as $\Delta\theta$ approaches 0, the line PQ approaches the tangent line to the circle at P so ϕ